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Numerical methods for analysis of plates on tensionless elastic foundations

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Abstract

A numerical methodology for analysis of plates resting on tensionless elastic foundations, described either by the Winkler model or as an elastic half-space, is presented in this paper. The contact surface is assumed unbonded and frictionless. The finite element method is used to discretize the plate and foundation. To overcome the difficulties in solving the plate–foundation equilibrium equations together with the inequality constraints due to the frictionless unilateral contact condition, a variational formulation equivalent to these equations is presented from which three alternative linear complementary problems (LCP) are derived and solved by Lemke's complementary pivoting algorithm. In the first formulation, the LCP variables are the plate displacements and the elastic foundation reaction, in the second, the LCP is derived in terms of the elastic foundation reaction and, in the third formulation, the variables are the elastic foundation displacements and the gap between the bodies. Once the LCP is solved the no-contact regions where the plate lifts up away from the foundation and the sub-grade reaction, as well as the plate displacements and stresses, can be easily obtained. The methodology is illustrated by three examples and the results are compared with existing analytical and numerical results found in the literature. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Plates; Elastic foundation; Contact problems; Unilateral contact

1. Introduction

This paper presents a numerical methodology for estimating the structural behavior of plates resting on tensionless elastic foundations. The analyses of plates on elastic foundations have been motivated by the need in engineering to design, for example, mat and raft foundations, pavement slabs of roads and airfield, floor systems of industrial yards and flexible column footings. These problems are usually analyzed by assuming that the foundation reacts in compression as well as in tension. However, in many practical situations, this assumption is questionable. Some supporting media cannot sometimes provide tensile reaction and, under certain conditions, some portions of the plate may lift-off. This is, for example, the case of

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plates resting on soils lacking both adhesive and cohesive properties. In these circumstances, if the unilateral character of the foundation is not taken into account, the engineer may incur considerable error, as shown in this paper.

The unbonded frictionless contact problem of plates resting on a tensionless foundation is complicated because the location and extent of the contact regions are not known at the outset. Since the stresses and deformations of the plate and foundation depend on the contact area and therefore on its unknown boundaries, these boundaries are, along with other mechanical quantities, part of the solution, being the primary unknowns of the problem. So, even for cases involving linear foundation models and linear plate theories, the problem is non-linear by virtue of unilateral constraints and therefore needs to be solved iteratively.

A critical step in the analysis of contact problems is the selection of a numerical methodology to deal with unilateral contact constraints. Basically there are three major numerical approaches for this problem, namely the Lagrange multiplier method, the penalty method and mathematical programming methods. This last alternative enables one to solve the contact problem by directly minimizing the potential energy containing explicitly moving boundary parameters and the associated inequality constraints and thus maintaining the original mathematical characteristics of the problem. Some of the optimization's techniques used for the contact problem are: linear and quadratic programming, recursive quadratic programming or, alternatively, methods for the solution of linear complementary problems such as Lemke's or Dantzig's algorithms.

The bending of plates resting on elastic foundations has been the subject of numerous investigations in the past. The first attempts to solve the problem of a plate on tensionless foundation include, among other, the works of Weitsman (1969, 1970). These works were followed by the contributions by Conry and Seireg (1971), Svec and McNeice (1972) and Svec (1974). Gladwell and Iyer (1974) studied the frictionless unilateral contact problem between an infinite half-space and a circular plate, subjected to its own weight plus a distributed load on a central circular area. Chand et al. (1976) formulated the unilateral contact problem between two elastic bodies as a quadratic programming problem and showed that the solution of a contact problem, if feasible, is unique and can be easily found by the modified simplex method of quadratic programming. Variational formulations for the solution of unilateral contact problems were discussed by Ascione and Grimaldi (1984) and results for a circular plate were presented. In one of the formulations proposed in this paper they arrive, after using the FEM to discretize the plate and foundation and Kuhn–Tucker conditions (Luenberger, 1973), at a linear complementary problem (LCP) which is solved through the use of Dantzig's algorithms (Cottle and Dantzig, 1968). Ascione and Olivito (1985) presented a formulation based on the penalty method to solve the unilateral contact problem between a rectangular plate and an infinite half-space. Rajapakse and Selvadurai (1986) made a comparative study on the efficiency of some plate finite elements in the analysis of plates resting on an infinite half-space.

In the eighties, works employing the boundary element method (BEM) for the analysis of contact problems began to appear. Puttonen and Varpasuo (1986) examined the applicability of the BEM to the analysis of plates on a Winkler or Pasternak foundation and Katsikadelis and Kallivokas (1986) developed a procedure for the analysis of slender plates submitted to different loading conditions in contact with a Pasternak foundation. More recently, Hu and Hartley (1994) used the BEM to analyze the behavior of thin plates on an elastic half-space.

Meanwhile, Li and Dempsey (1988) presented a solution for the frictionless unilateral contact problem between a square plate subjected to a vertical load and an infinite half-space or a Winkler foundation. Also in 1988, two analytical works on the frictionless unilateral contact problem appeared. In one of them Celep (1988), using Galerkin's method, studied the behavior of rectangular plates submitted to concentrated and distributed loads in contact with a frictionless and tensionless Winkler foundation. In the other Celep et al. (1988), analyzed the unilateral contact problem between a circular plate and an elastic foundation constituted by discreet springs, using Ritz's method.

More modern attempts to solve contact problems include the works of Björkman et al. (1995), who used sequential quadratic programming (SQP) for the study of geometrically non-linear frictionless contact problems, and Silveira and Gonçalves (1997), who presented a numerical methodology for the geometrically non-linear analysis of slender structural elements with unilateral constraints combining a linear complementary problem formulation with arc-length techniques. Also recently, Akbarov and Kocatürk (1997) used the Galerkin's method to study the bending of anisotropic plates on a tensionless foundation.

The present work adds a new contribution to this field by providing some alternative formulations for general-purpose analysis of plates resting unilaterally on an elastic foundation. In this analysis, the plate and the foundation are treated as two separated elastic bodies in unilateral contact at the interface. The plate is described either by Kirchhoff or Reissner–Mindlin's plate theory and the foundation is considered as a Winkler foundation or an elastic half-space. The plate and the foundation are discretized using the finite element method. To overcome the difficulties in solving the plate–foundation equilibrium equations together with the inequality constraints due to the frictionless unilateral contact condition, a variational formulation equivalent to these equations is presented from which three alternative LCP are derived and solved by Lemke's complementary pivoting algorithm. In the first formulation, the LCP variables are the plate displacements and the elastic foundation reaction, in the second, the LCP is derived in terms of the elastic foundation reaction and, in the third formulation, the variables are the elastic foundation displacements and the gap between the bodies. Once the LCP is solved the no-contact regions where the plate lifts up away from the foundation and the subgrade reaction, as well as the plate displacements and stresses, can be easily obtained. Numerical examples involving plates under different loading conditions are presented and the results are compared with existing results to demonstrate the validity and effectiveness of the formulations derived in this paper.

2. Problem formulation

Consider a structural system consisting of an elastic plate and a tensionless elastic foundation. The plate is defined as a solid elastic continuum which occupies a domain V, limited by three different regular surfaces: S_c, S_u and S_f, where S_u is the area where displacements are prescribed, S_f is the area where the surface forces are prescribed and S_c is the surface where contact may occur. For the plate, the equilibrium equations, the strain–displacement relations and the constitutive equations are given, respectively, by

$$\sigma_{ij,j} = 0, \tag{1}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{3}$$

where σ_{ij} are the Cauchy stress components, ε_{ij} are the infinitesimal strain components, u_i are the displacements and C_{ijkl} are the material parameters. In this paper the plate is analyzed by either the Kirchhoff or Reissner–Mindlin plate theory.

The elastic foundation is described by

$$r_b = C_b u_b \tag{4}$$

where u_b and r_b are the displacement and reaction of the elastic foundation, respectively, and C_b is the foundation modulus.

For the structural system studied here, the following boundary conditions must be satisfied.

$$u_i = \bar{u}_i \quad \text{on } S_u, \tag{5}$$

$$F_i = \sigma_{ij} n_j \quad \text{on } S_\mathrm{f}, \tag{6}$$

$$\varphi = u_\mathrm{b} - u_i \geqslant 0 \quad \text{on } S_\mathrm{c}, \tag{7}$$

where, u_i is the deflection of the plate orthogonal to the foundation and φ is the gap between the plate and the foundation in the potential contact region S_c. Inequality (7) is the compatibility condition that represents the impenetrability between the bodies. When $\varphi = 0$, there is contact and $r_\mathrm{b} \geqslant 0$. On the other hand, when there is no contact, $\varphi \geqslant 0$ and $r_\mathrm{b} = 0$.

So, the conditions that define in a complete way the contact as being unilateral are given by the inequality

$$r_\mathrm{b} \geqslant 0 \tag{8}$$

and, the following complementarity relationship between φ and r_b

$$\int_{S_\mathrm{c}} r_\mathrm{b} \varphi \, \mathrm{d}S_\mathrm{c} = 0. \tag{9}$$

In these equations compressive reactions are assumed to be positive.

The problem unknowns can be obtained by solving Eq. (1) together with boundary conditions (5) and (6), inequalities (7) and (8) and the complementarity condition (9). The non-linearity due to the unilateral constraints makes it difficult to solve the contact problem directly. For this reason, an equivalent minimization problem is formulated which is particularly suitable for numerical analysis. It can be shown that the optimization's problem (Joo and Kwak, 1986; Silveira, 1995)

$$\text{Min} : J(u, u_\mathrm{b}) \tag{10}$$

$$\text{s.t.} : -\varphi \leqslant 0, \;\; \text{on } S_\mathrm{c} \tag{11}$$

where,

$$J = \frac{1}{2} \int_V C_{ijkl} \varepsilon_{kl} \varepsilon_{ij} \, \mathrm{d}V + \frac{1}{2} \int_{S_\mathrm{c}} C_\mathrm{b} u_\mathrm{b}^2 \mathrm{d}S_\mathrm{c} - \int_{S_\mathrm{f}} F_i u_i \, \mathrm{d}S_\mathrm{f} \tag{12}$$

is equivalent to the contact problem described above.

Based on these equations, three alternative LCP are proposed for the numerical analysis of plates resting on a tensionless foundation in the following sections.

2.1. Formulation 1

According to Ascione and Grimaldi (1984), restrictions (7)–(9) can be substituted by the variational inequality

$$\int_{S_\mathrm{c}} \sigma \varphi \, \mathrm{d}S_\mathrm{c} \geqslant 0, \tag{13}$$

where σ belongs to the positive cone $\bar{\boldsymbol{K}}$, in which the admissible reactions r_b are the elements

$$\bar{\boldsymbol{K}} = \left\{ r_\mathrm{b} \;\; \in \boldsymbol{Y}', \int_{S_\mathrm{c}} r_\mathrm{b} w \, \mathrm{d}S_\mathrm{c} \geqslant 0, \;\; \forall w \in \boldsymbol{Y}, \;\; w \geqslant 0 \right\} \tag{14}$$

and $\boldsymbol{Y}'$ and $\boldsymbol{Y}$ are the vectorial spaces that contain the solutions to r_b and φ, respectively. The complementary condition (9) is satisfied when $\sigma = r_\mathrm{b}$.

Then, the contact constraint can be eliminated from the analysis, by writing

$$J_1 = \frac{1}{2}\int_V C_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\,\mathrm{d}V + \frac{1}{2}\int_{S_c} C_b u_b^2\,\mathrm{d}S_c - \int_{S_c} r_b\varphi\,\mathrm{d}S_c - \int_{S_f} F_i u_i\,\mathrm{d}S_f. \tag{15}$$

The first variation of J_1, after eliminating u_b from the previous equation by way of relation (7), is given by the following variational inequality (see Eq. (13)):

$$\delta J_1 = \int_V C_{ijkl}\varepsilon_{kl}\,\delta\varepsilon_{ij}\,\mathrm{d}V + \int_{S_c} C_b(\varphi + u)\,\delta u\,\mathrm{d}S_c + \int_{S_c}[C_b(\varphi+u) - r_b]\delta\varphi\,\mathrm{d}S_c - \int_{S_c}\varphi\,\delta r_b\,\mathrm{d}S_c - \int_{S_f} F_i\delta u_i\,\mathrm{d}S_f \leqslant 0. \tag{16}$$

Elimination of φ from the Eq. (16), by use of Eqs. (4) and (7), leads to a variational inequality in terms of u, e, r_b only, which corresponds to the first variation of the following integral:

$$J_1 = \frac{1}{2}\int_V C_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\,\mathrm{d}V - \frac{1}{2}\int_{S_c} D_b r_b^2\,\mathrm{d}S_c + \int_{S_c} r_b u\,\mathrm{d}S_c - \int_{S_f} F_i u_i\,\mathrm{d}S_f, \tag{17}$$

where $D_b = C_b^{-1}$. The variables u, e, r_b must be obtained so that the first variation of the functional J_1 satisfies the inequality $\delta J_1(u, r_b) \leqslant 0$.

Now, using the finite element method, one can assume that, for a generic plate element, the displacement field within the element, u, is related to the nodal displacements $\mathbf{u}$ by

$$u = \mathbf{N}\mathbf{u}, \tag{18}$$

where $\mathbf{N}$ is the usual interpolation functions matrix. For a generic elastic foundation element, the compressive reaction is related to its nodal values by

$$r_b = \mathbf{H}_b r_b, \tag{19}$$

where $\mathbf{H}_b$ is the matrix that contains the interpolation functions that describe the behavior of the elastic base.

From these definitions and adding the contributions of each finite element, one arrives at the discretized functional of the problem in the global form

$$\bar{J}_1 = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} - \frac{1}{2}r_b^{\mathrm{T}}\mathbf{T}r_b + r_b^{\mathrm{T}}\mathbf{A}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{R} \tag{20}$$

where $\mathbf{R}$ is the nodal load vector, $\mathbf{K}$ is the stiffness matrix, $\mathbf{A}$ is the joining matrix between the structure and the elastic foundation, defined by

$$\mathbf{A} = \sum_{m_c}\int_{S_c} \mathbf{H}_b^{\mathrm{T}}\mathbf{N}\,\mathrm{d}S_c \tag{21}$$

and $\mathbf{T}$ is the flexibility matrix of the elastic foundation which can be written as

$$\mathbf{T} = \sum_{m_c}\int_{S_c} \mathbf{H}_b^{\mathrm{T}} D_b \mathbf{H}_b\,\mathrm{d}S_c. \tag{22}$$

Here m_c is the number of elements of the contact region.

After the first variation of Eq. (20), one arrives at the following LCP in terms of the plate displacements and foundation reaction (Ascione and Grimaldi, 1984; Silveira, 1995):

$$\mathbf{K}\mathbf{u} + \mathbf{A}^{\mathrm{T}}r_b - \mathbf{R} = \mathbf{0}, \tag{23}$$

$$\mathbf{A}\mathbf{u} - \mathbf{T}r_b \leqslant \mathbf{0}, \tag{24a}$$

$$r_b \geqslant \mathbf{0}, \tag{24b}$$

$$(\mathbf{Au} - \mathbf{T}r_b)^{\mathrm{T}} r_b = \mathbf{0}. \tag{24c}$$

The solution of the Eq. (23), considering the constraints (24), can be achieved through the use of mathematical programming methods, in particular, pivoting techniques developed for complementary problems (Cottle and Dantzig, 1968; Lemke, 1968). However, first it is necessary to reduce the previous relations to a standard LCP form. This can be obtained through the use of the following definitions:

$$\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-; \quad \mathbf{z}_1 = \mathbf{T}r_b - \mathbf{Au}; \quad \mathbf{z}_2 = \mathbf{Ku} + \mathbf{A}^{\mathrm{T}} r_b - \mathbf{R}; \quad \mathbf{z}_3 = -\mathbf{z}_2, \tag{25}$$

where $\mathbf{u}^+ \geqslant \mathbf{0}$, $\mathbf{u}^- \geqslant \mathbf{0}$ are the positive and negative parts of the vector $\mathbf{u}$ (Fletcher, 1981).

Using these new variables, it is possible to write Eqs. (23) and 24(a–c) in the following form:

$$\mathbf{w} = \mathbf{q} + \mathbf{Mz}, \tag{26}$$

$$\mathbf{w} \geqslant \mathbf{0}, \tag{27a}$$

$$\mathbf{z} \geqslant \mathbf{0}, \tag{27b}$$

$$\mathbf{w}^{\mathrm{T}}\mathbf{z} = \mathbf{0} \tag{27c}$$

with

$$\mathbf{M} = \begin{bmatrix} \mathbf{K} & -\mathbf{K} & \mathbf{A}^{\mathrm{T}} \\ -\mathbf{K} & \mathbf{K} & -\mathbf{A}^{\mathrm{T}} \\ -\mathbf{A} & \mathbf{A} & \mathbf{T} \end{bmatrix}, \quad \mathbf{q} = \begin{Bmatrix} -\mathbf{R} \\ \mathbf{R} \\ \mathbf{0} \end{Bmatrix}, \quad \mathbf{z} = \begin{Bmatrix} \mathbf{u}^+ \\ \mathbf{u}^- \\ \mathbf{r}_b \end{Bmatrix} e\mathbf{w} = \begin{Bmatrix} \mathbf{z}_2 \\ \mathbf{z}_3 \\ \mathbf{z}_1 \end{Bmatrix}. \tag{28}$$

Eq. (26) and constraints (27) correspond to a standard linear complementary problem which is solved here by Lemke's algorithm.

2.2. Formulation 2

If the stiffness matrix in Eq. (23) is positive definite, it is possible to establish the following relationship between $\mathbf{u}$ and r_b:

$$\mathbf{u} = \mathbf{K}^{-1}(\mathbf{R} - \mathbf{A}^{\mathrm{T}} r_b). \tag{29}$$

Substituting Eq. (29) in Eq. (23), one arrives at a variational expression that is a function of the nodal values of the base reaction r_b only

$$\bar{J}_2 = -\tfrac{1}{2} r_b^{\mathrm{T}} \mathbf{P} r_b + r_b^{\mathrm{T}} \mathbf{H} - s. \tag{30}$$

Here $\mathbf{P}$ is a symmetric positive definite matrix, $\mathbf{H}$ is a vector and s is a constant which are defined as

$$\mathbf{P} = \mathbf{AK}^{-1}\mathbf{A}^{\mathrm{T}} + \mathbf{T}, \quad \mathbf{H} = \mathbf{AK}^{-1}\mathbf{R}, \quad s = \tfrac{1}{2}\mathbf{R}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{R}. \tag{31}$$

Eq. (30), with the foundation reaction constraint condition, characterizes a quadratic programming problem. Again one can derive from this formulation a standard LCP similar to the one described by Eqs. (26) and (27), considering now, $\mathbf{M} = \mathbf{P}$, $\mathbf{q} = -\mathbf{H}$, $\mathbf{z} = r_b$ and $\mathbf{w}$ as the Lagrange multiplier introduced in the analysis to represent the impenetrability condition between the bodies. After the calculation of r_b, $\mathbf{u}$ can be obtained from Eq. (29).

2.3. Formulation 3

In this third formulation the LCP is written in terms of the nodal variables in the contact region, that is, the base displacements and the gap between the two bodies.

Discretizing Eq. (12) by the use of the FEM and adding the contributions of each finite element, one arrives at the discretized functional of the problem in the global form

$$J_1 = \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} + \tfrac{1}{2}\mathbf{u}_{\mathrm{b}}^{\mathrm{T}}\mathbf{K}_{\mathrm{b}}\mathbf{u}_{\mathrm{b}} - \mathbf{u}^{\mathrm{T}}\mathbf{R}. \tag{32}$$

The first variation of Eq. (32) with respect to the nodal values of $\mathbf{u}$ and $\mathbf{u}_{\mathrm{b}}$, leads to the equilibrium equation

$$\mathbf{K}\mathbf{u} + \mathbf{K}_{\mathrm{b}}\mathbf{u}_{\mathrm{b}} = \mathbf{R}. \tag{33}$$

After some algebraic manipulation, Eq. (33) can be rewritten as

$$\mathbf{w} = \mathbf{q} + \mathbf{M}\mathbf{z}, \tag{34}$$

where $\mathbf{M} = (\mathbf{K} + \mathbf{K}_{\mathrm{b}})^{-1}\mathbf{K};\ \mathbf{q} = (\mathbf{K} + \mathbf{K}_{\mathrm{b}})^{-1}\mathbf{R};\ \mathbf{z} = \mathbf{u}_{\mathrm{b}} - \mathbf{u}$ and $\mathbf{w} = \mathbf{u}_{\mathrm{b}}$.

Eq. (34) together with the constraints

$$\mathbf{w} \geqslant \mathbf{0}, \tag{35a}$$

$$\mathbf{z} \geqslant \mathbf{0}, \tag{35b}$$

$$\mathbf{w}^{\mathrm{T}}\mathbf{z} = \mathbf{0}, \tag{35c}$$

defines again a standard LCP.

It should be pointed out that the order of the matrices and vectors in formulations 2 and 3 are practically the same. But, in order to obtain Eqs. (34) and (35) substructuring techniques and static condensation are necessary.

3. Examples

In this section three examples are presented in order to verify the efficiency and reliability of the formulations developed in this work.

3.1. Supported continuous beam on a Winkler tensionless elastic foundation

The first numerical example, used to test formulations 1–3, is shown in Fig. 1(a). It consists of a simply supported beam, modeled as a long plate resting on a Winkler foundation, and subjected to concentrated moments ($M = 100$) on the edges. Due to the loading conditions, a non-contact region will appear near the right edge.

To model the structural system (plate and foundation) 20 isoparametric finite elements with eight nodes each were used as shown in Fig. 1(b). The plate elements were derived from the Reissner–Mindlin theory. The elastic foundation reaction is interpolated using the nodal points of the vertices of the element. The beam dimensions are: $a = 10$, $b = 1.0$ and $t = 0.4$, the Young's modulus and Poisson's ratio of the plate material are, respectively, 10^6 and 0.0 and, the foundation stiffness parameter K is taken as 71.68 (flexible foundation).

Figs. 2 and 3 show, respectively, the variation of the deflection and base reaction along the beam axis. As observed, all formulations give practically the same results and they are in good agreement with the

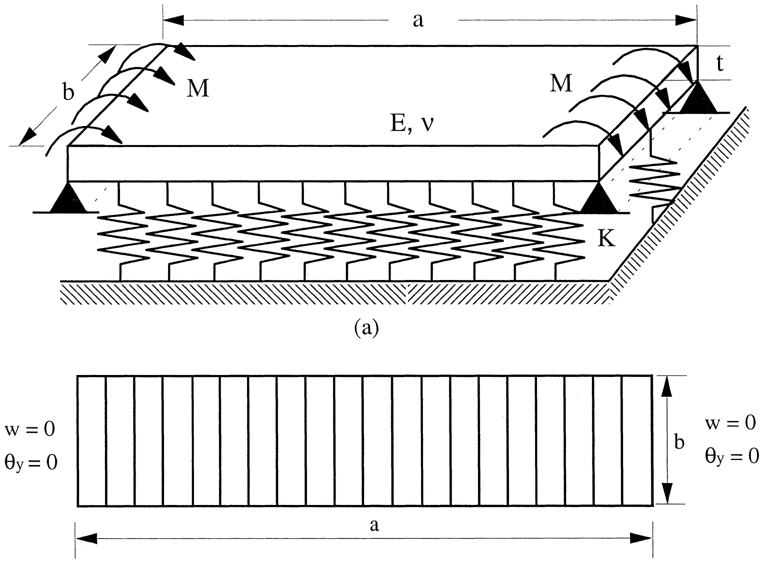

Fig. 1. (a) Continuum plate on Winkler tensionless elastic foundation and (b) structural system model.

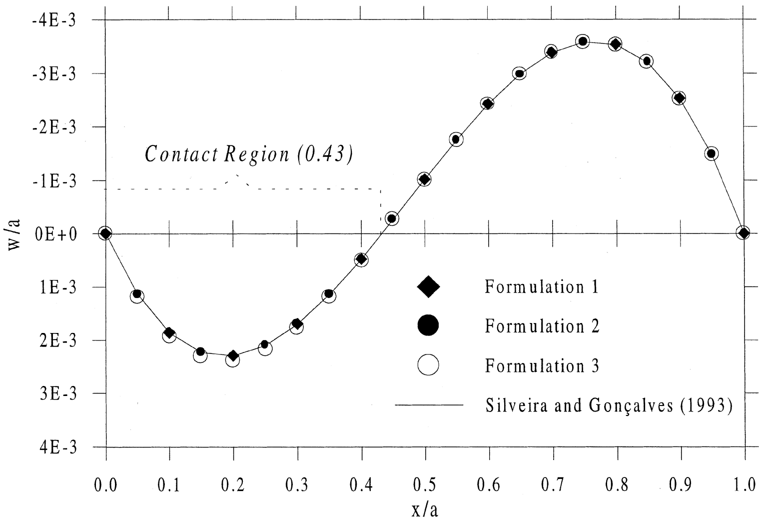

Fig. 2. Deformed shape of the plate.

analytical modal solution obtained by Silveira and Gonçalves (1993) specifically for this problem. To compare the efficiency of the proposed formulations, the computing time for each LCP is shown in Table 1 together with the order of matrix **M** in each LCP. Clearly formulations 2 and 3 are more efficient in terms of computing time than formulation (1), having approximately the same performance. One of the main reasons, as observed in Table 1, is the order of the matrices in each formulation. In the other numerical experiments presented in this paper, similar results were observed. On the other hand the second formulation seems to be more robust numerically and no numerical problem was experienced in this work when

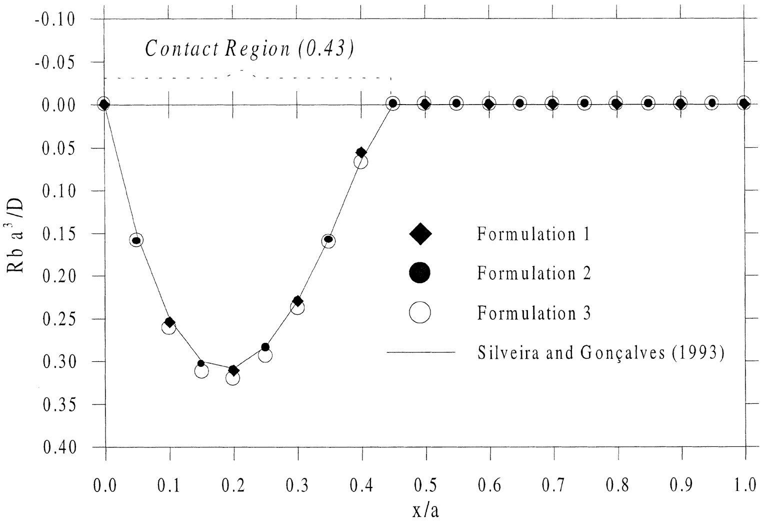

Fig. 3. Contact pressure distribution.

Table 1
Computing times for solving each LCP and matrix size

	Formulation 1	Formulation 2	Formulation 3
Time (s)	126	9	3
Order of matrix **M**	632 × 632	38 × 38	38 × 38

solving PLC-2 by Lemke's algorithm. In the other two cases in some experiments numerical difficulties were observed. These difficulties were mainly due to the sensitivity of Lemke's algorithm to the relative magnitude of the elements of matrix **M**. A possible solution is to choose the problems units in such a way that all elements in the LCP are of comparable magnitude.

The influence of the foundation stiffness parameter K on the behaviour of the plate is now analyzed. In Fig. 4 the deflection along the beam axis is given for different values of the foundation modulus. As one can observe, the area of contact region (and the corresponding displacements) decreases steadily as K increases, while the displacements of the non-contact region increases. The dependence of the contact area on the foundation stiffness is one of the main characteristics of tensionless foundation as compared with the conventional foundation.

Finally, Fig. 5 compares the results for the lateral deflection of a plate on a foundation with $K = 7168$ with those obtained for the same plate without foundation and with a foundation that reacts in compression as well as in tension. These results show clearly that there is for relatively stiff foundations a marked difference between the displacements of the tensionless and the conventional foundation models. As a consequence, considerable error may result if the unilateral character of the foundation is not taken into account in the analysis.

3.2. Slender plate on an elastic half-space

In this example a Kirchhoff square plate freely resting on an infinite half-space is considered. The structural model is shown in Fig. 6(a). For comparison purposes, in the presentation of the results, the following foundation/plate relative stiffness parameter

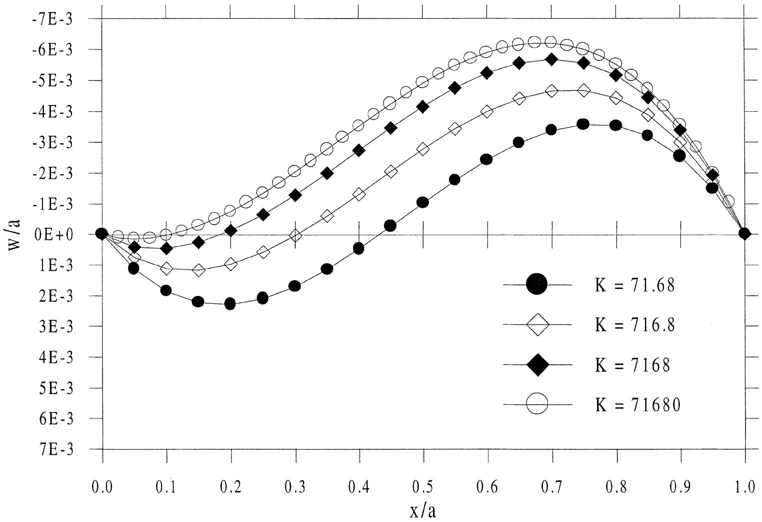

Fig. 4. Deformed profiles of the plate for several values of the foundation stiffness *K*.

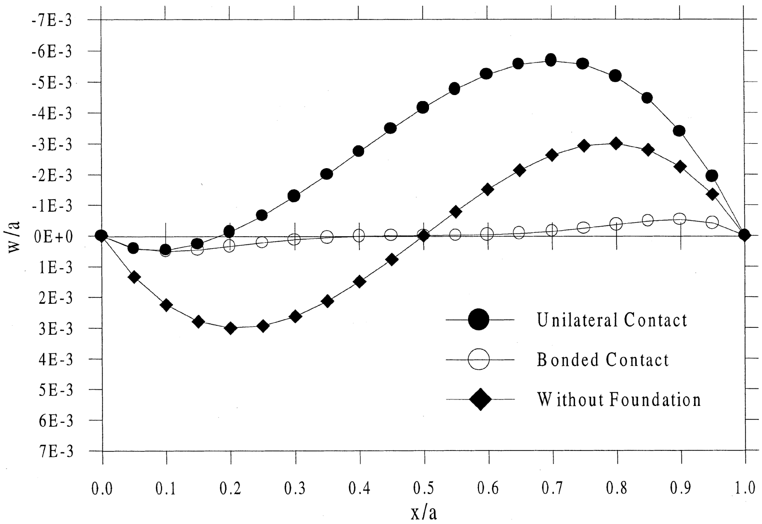

Fig. 5. Plate deformation pattern ($K = 7168$).

$$\gamma = \frac{\pi E_b a^3}{D(1 - \nu_b^2)} \tag{36}$$

defined by Gorbunov-Possadov and Serebrjanyi (1961) for a rectangular plate and used by Hu and Hartley (1994), is used here.

According to these authors, a square plate can be considered rigid when $\gamma \leqslant 8$.

The associated displacement parameter given by these authors for a rigid square plate is

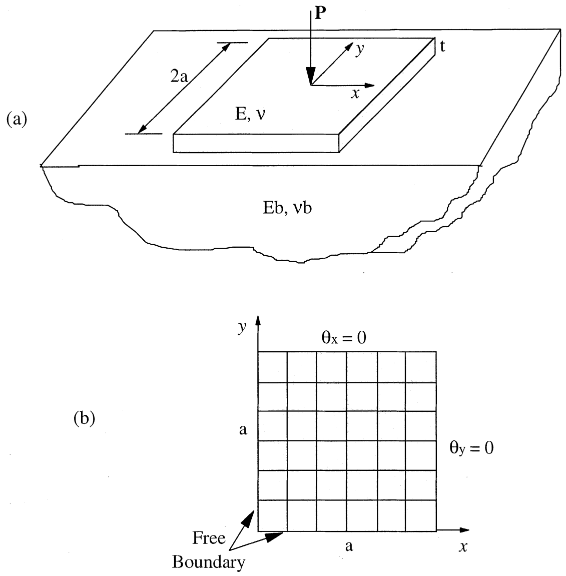

Fig. 6. (a) Slender plate on an elastic half-space and (b) system structural model.

$$W_{\mathrm{r}} = \frac{0.913(1 - \nu_{\mathrm{b}}^2)P}{2aE_{\mathrm{b}}} \tag{37}$$

where P is the magnitude of the total vertical load.

In this example, the structural system is modeled with 36 finite elements as shown in Fig. 6(b). Due to symmetry only 1/4 of the plate is analyzed. The value of the Poisson's ratio for both plate and the half-space is taken as 0.15.

First a square plate subjected to a uniformly distributed load q is analyzed and the results are compared with those obtained by Hu and Hartley (1994) using the boundary element method. In this case P in Eq. (37) is the resultant of the distributed load ($4a^2q$). In Fig. 7 the variation of the deflection of the plate along the symmetry axis x is shown for selected values of the stiffness parameter γ. Here the deflection is normalized with respect to W_{r} and the coordinate x is divided by a. As observed by Hu and Hartley (1994), the plate displacement decreases as γ decreases and approaches a straight line for low values of γ (nearly rigid plate or soft foundation) and the plate moves vertically as a rigid body with w practically equal to the reference value W_{r}. For stiffer foundations or more flexible plates (large γ) the plate deforms noticeably. For any value of γ the plate is completely in contact with the elastic foundation and the displacement at the boundary ($x/a = \pm 1$) tends to the reference value W_{r}. The corresponding distribution of the contact pressure along the plate central axis for each value of the stiffness parameter γ is shown in Fig. 8. Finally in Table 2 the value of the moment M_x in the plate center is presented for selected values of γ. In all the cases analyzed here the results compare well with those obtain by Hu and Hartley (1994).

In Figs. 9 and 10 and in Table 3, the results for central deflection, base reaction and moment M_y along the x axis on the plate edge are shown for increasing values of the stiffness parameter γ for a plate under a concentrated load acting on the center of the plate. The results are for the central axis of the plate. These results are again compared with the numerical results obtained by Hu and Hartley (1994).

As observed if Fig. 10, $\gamma = 0.03$ correspond practically to a rigid plate. As γ increases, the base reaction increases in the central area and when γ approaches 300 the plate edge loses contact with the foundation.

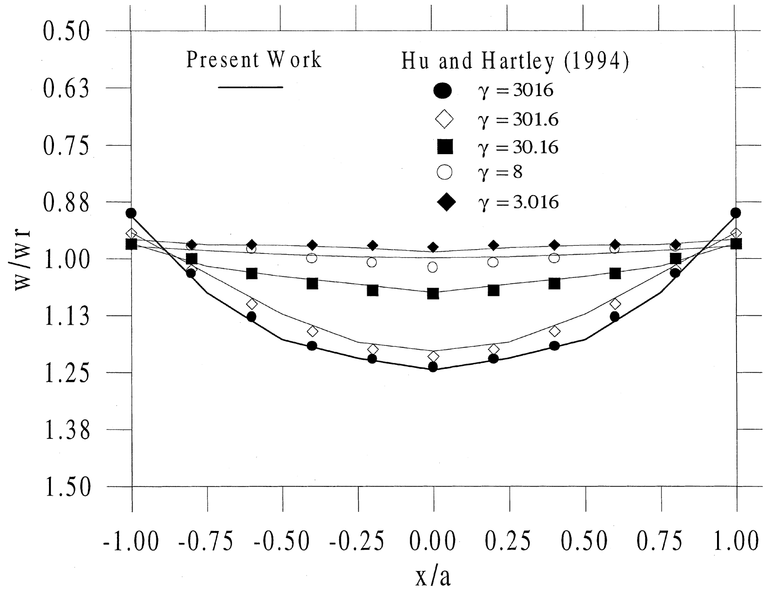

Fig. 7. Deflections along the centerline of the uniformly loaded plate.

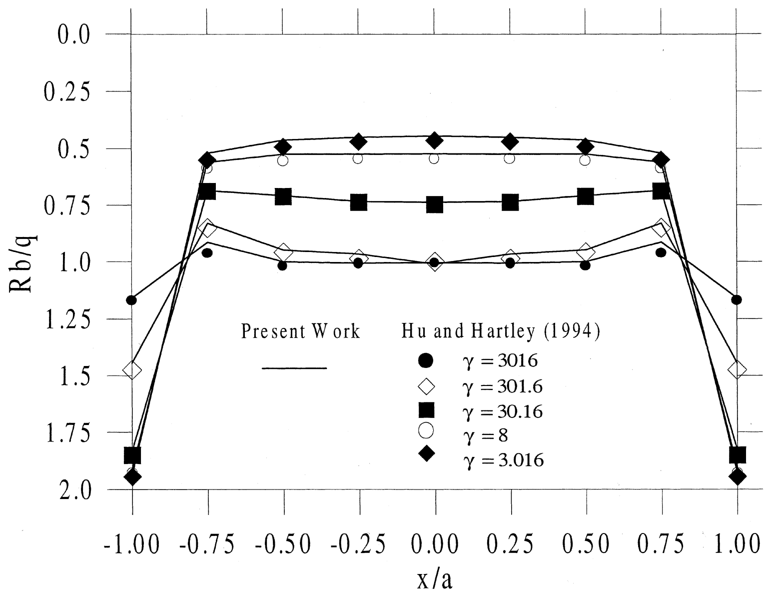

Fig. 8. Distribution of the foundation reaction – uniformly distributed load.

For higher values of γ the contact region decreases steadily and, since the applied load remains the same, in order to maintain the equilibrium, the foundation reaction increases substantially. At the same time, the uplift along the edges becomes more pronounced. Again the results illustrates the marked difference between the bonded and unbonded foundation models.

Table 2
Moments in the plate center – distributed load q

γ	M_x/P	
	Present work	Hu and Hartley (1994)
3.016	0.0221	0.0215
30.16	0.0147	0.0143
3016	0.0002	0.0002

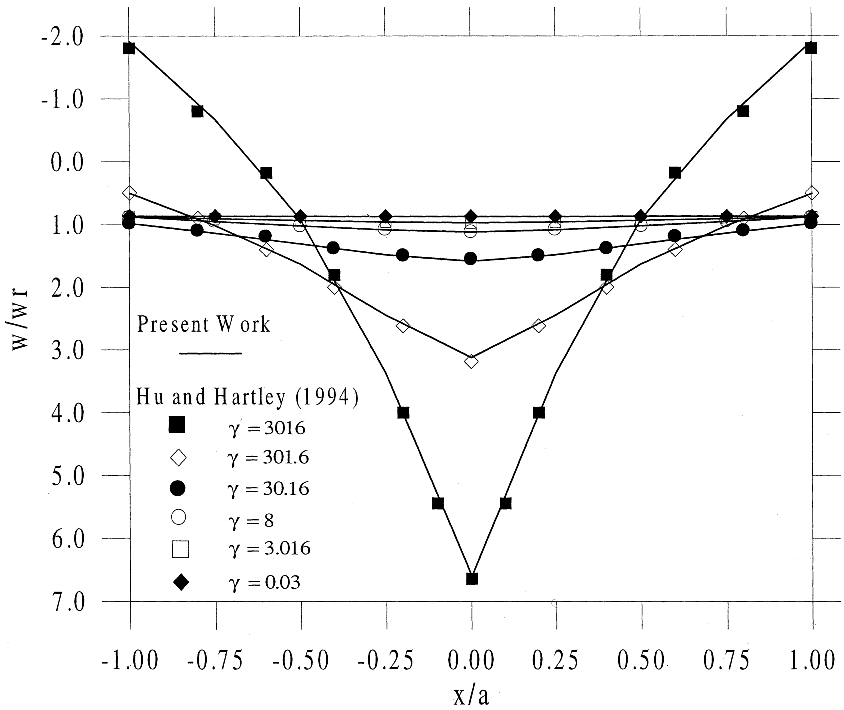

Fig. 9. Displacement profiles for different values of stiffness parameter γ. Plate under concentrated load.

3.3. Rectangular plates resting on Winkler tensionless elastic foundation

Fig. 11(a) shows the geometry of the plate used in this example. It is a square plate of flexural stiffness D with free boundaries and in contact with a tensionless Winkler foundation of stiffness K. The plate is submitted to a concentrated load acting at a generic point (X_p, Y_p). For comparison purposes, the following dimensionless quantities are used in the example:

$$k = \frac{K_a^4}{D}, \quad p = \frac{P_a}{D}, \quad x_p = \frac{X_p}{a}, \quad y_p = \frac{Y_p}{a}. \tag{38}$$

The results in Figs. 12 and 13 were obtained using the finite element mesh (half of the plate, only) presented in Fig. 11(b). Fifty isoparametric finite elements with eight nodal points and Mindlin's plate theory were used in the analysis. In Fig. 11(b) are also shown the adopted boundary conditions. These

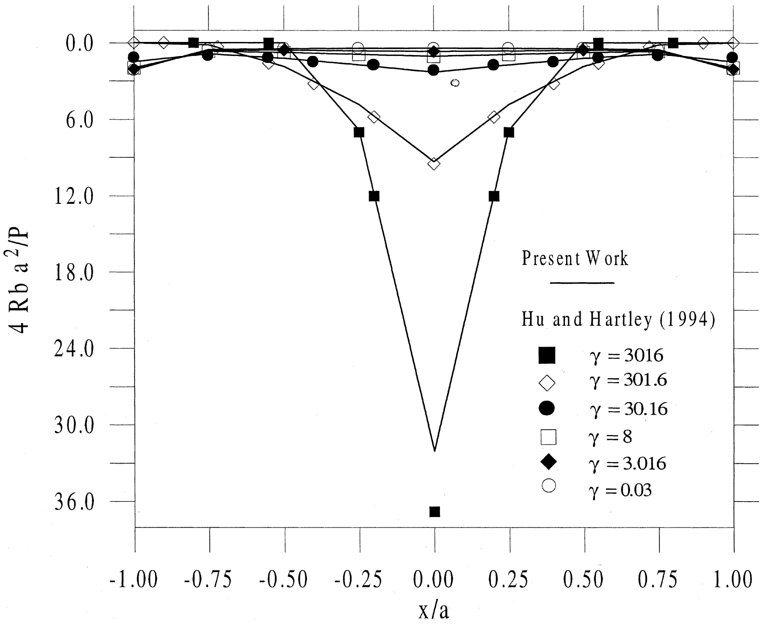

Fig. 10. Base reaction – concentrated load.

Table 3
Moment M_y Acting on the plate edge (x axis) – concentrated load

γ	M_y/P	
	Present work	Hu and Hartley (1994)
3.016	0.073	0.075
30.16	0.042	0.043
3016	0.012	0.012

results are compared with those obtained by Celep (1988) using the Galerkin method. The plate geometry and material properties are identical to those given by Celep: $K = 1000$ and $\nu = 0.25$ and, also, $E = 10$, $t = 0.018$ and $2a = 1.0$.

Fig. 12 shows the curves that separates the contact and the non-contact regions of the plate subjected to a concentrated load at different x_p positions and $y_p = 0.0$ while Fig. 13 shows the variation of the deflection of the plate along the x axis for the same load positions. When the load is close to the center of the plate ($x_p = 0$) the contact curve is practically a circle having a non-dimensional radius $r = 0.48$, that is close to the result found by Celep and, also by Akbarov and Kocatürk (1997) ($r = 0.49$). It was also found for the non-dimensional relationship $k^{1/4}r = 2.72$. This is also close to the value obtained by Celep ($k^{1/4}r = 2.76$). With the Winkler foundation replaced by an elastic half space, Weitsman (1970) obtained for a weightless plate of infinite extent $k^{1/4}r = 2.85$.

As observed in Figs. 12 and 14(a–d), the contact curve experiences a translation as the load moves away from the center of the plate along the x axis. Close to the edge of the plate, the contact area changes its shape and decreases rapidly, while the lift-off displacements increase at a similar rate.

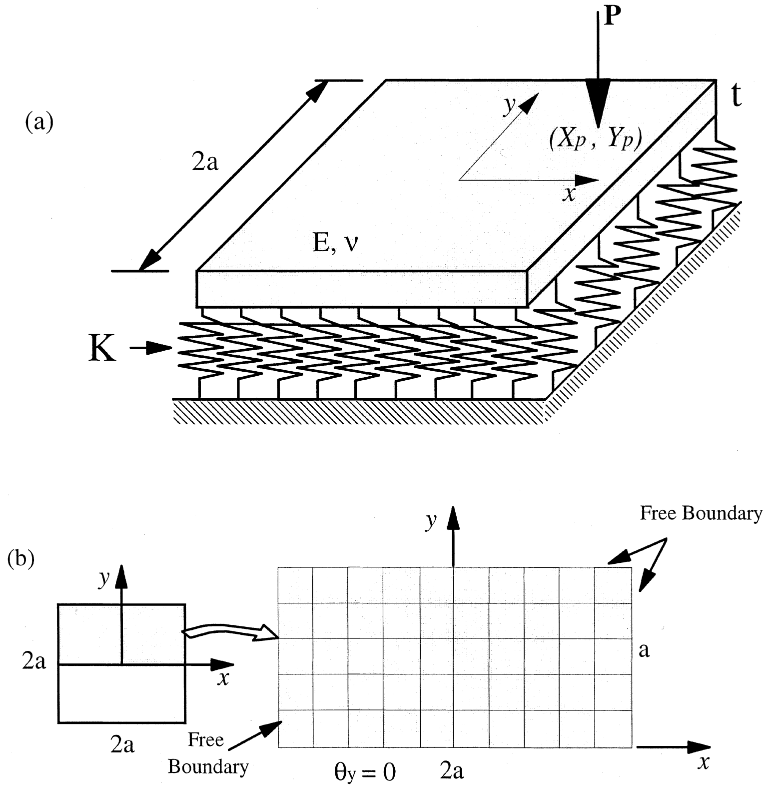

Fig. 11. (a) Rectangular plates resting on tensionless elastic foundation and (b) structural system model.

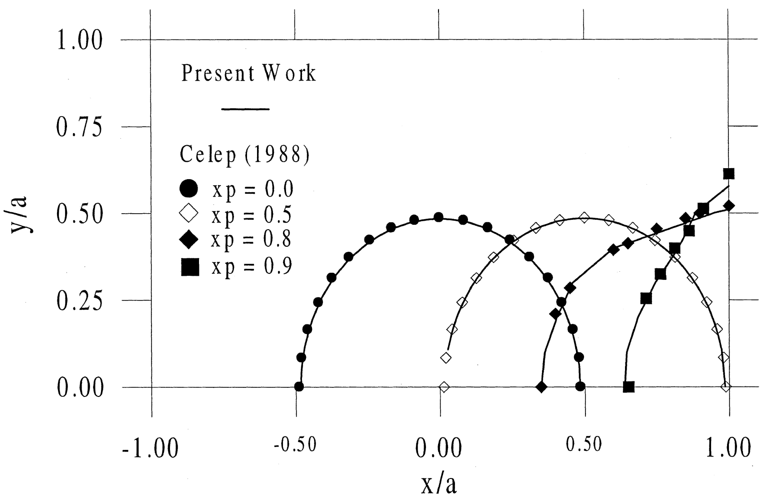

Fig. 12. Boundary curves that separate the contact and non-contact regions.

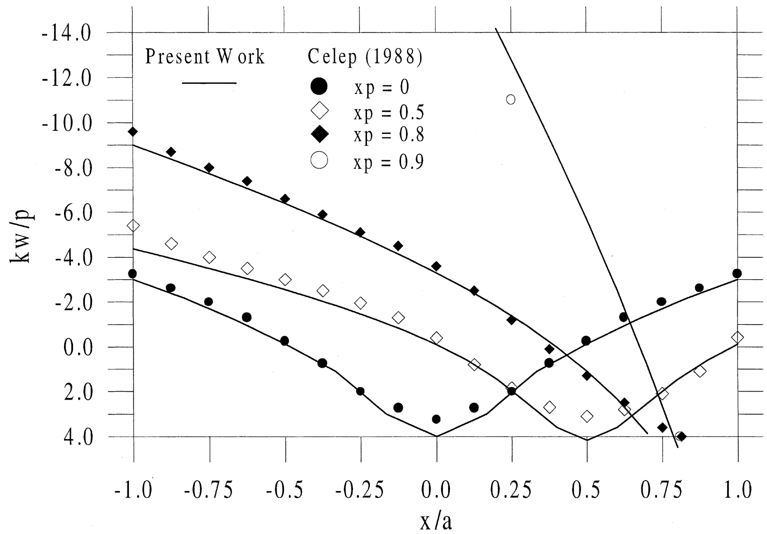

Fig. 13. Deformed profiles of the plate for different load positions along the x axis ($y_p = 0$).

4. Conclusions

The equilibrium analysis of the plates with unilateral contact constraints was studied in this paper by the use of the finite element method together with mathematical programming techniques. The use of mathematical programming techniques seems to be an efficient form of treatment of the unilateral constraints, since it allows the contact problem to be treated directly as a minimization problem involving the energy functional and the original variables of the problem with inequality constraints. Based on this variational formulation, three alternative linear complementary problems were formulated for the numerical analysis of plates resting on an elastic foundation: in the first formulation, the LCP variables are the plate displacements and the elastic foundation reaction, in the second, the LCP is written in terms of the elastic foundation reaction and, in the third formulation, the variables are the elastic foundation displacements and the gap between the bodies. These LCP were solved by the use of Lemke's algorithm. A comprehensive parametric study of the dominant parameters (foundation stiffness, sub-grade reaction, contact region, etc.) was carried out through numerical examples. The obtained results compare well with some analytical and numerical results found in literature. These examples validate the formulations and proposed numerical methodologies and clarifies the main characteristics of this type of structural problem. These results also show that for this class of problems considerable error may result if the unilateral character of the foundation is not taken into account in the analysis. When a partial contact occurs, the displacements in the lift-off region are larger in this case, while, in order to maintain equilibrium, the reaction in the contact region increases substantially, causing a marked difference between the displacements of the tensionless and the conventional foundation models.

Acknowledgements

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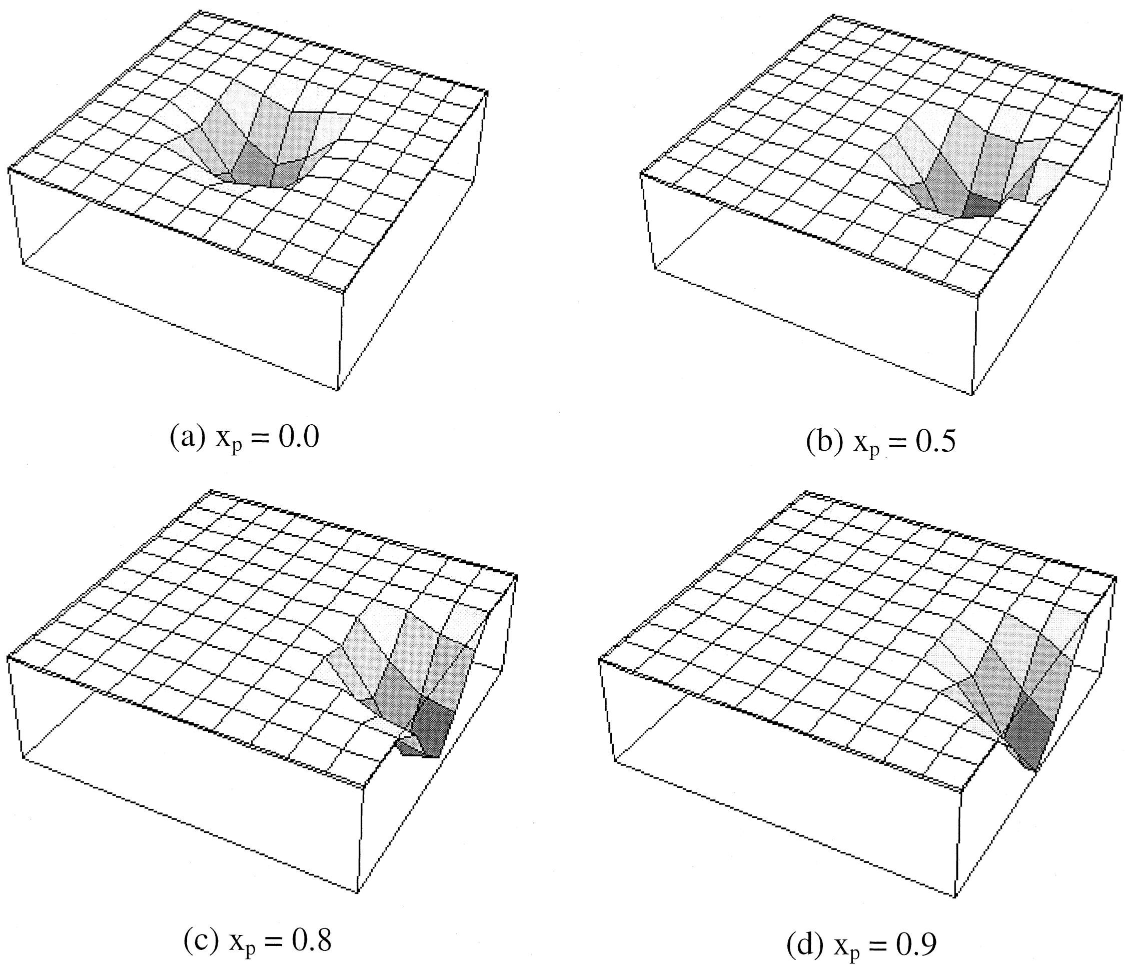

Fig. 14. Elastic foundation displacement for different load locations.

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